Tutorial 3

Exercise 1. In the Set Splitting problem, we are given a family of sets $\mathcal{F}$ over a universe $\mathcal{U}$ and a positive integer k, and the goal is to test whether there exists a coloring of $\mathcal{U}$ with two colors such that at least k sets in $\mathcal{F}$ are **non**-monochromatic. Show that the problem admits a kernel with at most $2k$ sets and $\mathcal{O}(k^2)$ universe size.
We say that the coloring that witnesses a `Yes` instance *splits at least k sets*.

Exercise 2. In the Minimum Maximal Matching problem, we are given an undirected graph G and a positive integer k, and the task is to decide whether there exists a *(inclusion-wise)* maximal matching in G on at most k edges. Find a polynomial kernel for the problem (parameterized by k).

Exercise 3. In the Connected Vertex Cover problem, we are given an undirected graph G and a positive k. The objective is to decide whether there exists a vertex cover C of G such that $|C| \leq k$ and $G[C]$ is connected.

- Show that the problem admits a kernel with at most $2^k + \mathcal{O}(k^2)$ vertices.
- Show that if G does not contain a cycle of length 4 as a subgraph, then the problem admits a kernel of size $\mathcal{O}(k^2)$.

Exercise 4. *(bounded branching for vertex cover)*

Observe that

1. for each vertex v in a graph, a valid vertex cover must contain either v or all its neighbors, and
2. if a graph has maximum degree $\Delta(G) = 1$, solving Vertex Cover is trivial for this graph.

Use these observations to obtain an $\mathcal{O}^*(\varphi^k)-$time algorithm for Vertex Cover, where φ is the golden ratio.

Now try to improve this approach. What running time can you achieve?